UDC 539.3

SIMPLE WAVES AND COLLAPSE OF A DISCONTINUITY IN AN ELASTIC-PLASTIC MEDIUM WITH MISES CONDITION

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It was shown in [1] that strong discontinuities different from the elastic shock waves may form in elastic-plastic media, and this fact leads to the necessity of formulating additional conditions at the discontinuities. It would therefore be desirable to be able to single out those media for which solutions could be constructed without the need of using such discontinuities.

In [2] it was shown that with the Mises yield condition and the isotropic work-hardening property, the plane simple waves do not break up in the presence of two velocity components when certain restrictions are imposed on the initial state of stress. This is valid for those media which, on a simple compression test, produce a stress-strain curve convex in the direction of the stress axis (Fig. 1). Below we generalize this assertion to the case of arbitrary plane simple waves in the same medium. A solution of the problem of collapse of an arbitrary discontinuity is constructed under stricter conditions. In this case the only discontinuities are the elastic shock waves, the contact type discontinuities and the discontinuities representing the limiting case of simple waves propagating at constant speed.

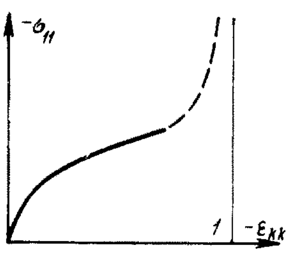

Fig. 1.

1. Using the framework of the geometrically linear theory, let us consider the motion of an elastic-plastic medium, the free energy of which is given by $F = F_1(\varepsilon_{ij}^e) + F_2(T)$ Under the usual assumptions [3], the temperature T does not enter the stress-strain equations. Consequently, for such a medium the mechanical problem can be solved separately from the heat problem. In particular, the condition of conservation of energy at the discontinuity does not impose any restrictions on the velocity and the stresses. It merely represents a boundary condition of the problem on the temperature distribution.

The equation of the stress surface written in the Mises form is as follows

$$J^2 \equiv {}^1/_2\, \sigma_{ij}'\sigma_{ij}' = k^2(\chi) \quad (d\chi = \sigma_{ij}d\varepsilon_{ij}^p \quad \text{or} \quad d\chi = \sqrt{d\varepsilon_{ij}^p\ d\varepsilon_{ij}^p}) \tag{1.1}$$

Here σ_{ij}' is the deviator of the stress tensor and $k^2(\chi)$ is a prescribed, monotonously increasing function. The relations within the parentheses are equivalent to each other for the condition (1.1) in the presence of an associated law.

The following rule associated with (1.1) is adopted for the plastic deformation increments

$$d\varepsilon_{ij}^p = d\lambda\sigma^{ij}, \qquad d\lambda \geqslant 0 \tag{1.2}$$

and the Hooke's law for the other deformations. Finally, for the total deformations we have

$$-K\frac{\partial v_k}{\partial x_k} = \frac{\partial p}{\partial t} \qquad (p = -{}^1/_3\,\sigma_{kk})$$

$$\frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right) - \frac{1}{3}\frac{\partial v_k}{\partial x_k}\delta_{ij} = \frac{\partial \lambda}{\partial t}\sigma_{ij}' + \frac{1}{2\mu}\frac{\partial \sigma_{ij}'}{\partial t} \quad (\mu,\ K = \text{const}) \tag{1.3}$$

From (1.1) and (1.2) we have $d\chi = 2k^2(\chi)d\lambda$. Thus $\chi = \chi(\lambda)$ and we can write (1.1) as $\lambda = \Phi(J)$ where ($\Phi(J)$ is a monotonously increasing function with the load condition

$$d\lambda = \begin{cases} \dfrac{d\Phi}{dJ}dJ & \text{when } \lambda = \Phi(J),\ dJ > 0 \\ 0 & \text{when } dJ \leqslant 0 \quad \text{and also when} \quad \Phi(J) < \lambda \end{cases} \tag{1.4}$$

Below we shall study plane waves propagating in a constant state, for (1.3), (1.4) and the equations of motion

$$\rho_0\frac{\partial v_1}{\partial t} = \frac{\partial \sigma_{11}}{\partial x}, \qquad \rho_0\frac{\partial v_2}{\partial t} = \frac{\partial \sigma_{12}}{\partial x}, \qquad \rho_0\frac{\partial v_3}{\partial t} = \frac{\partial \sigma_{13}}{\partial x} \tag{1.5}$$

In the case of a one-dimensional motion it follows from (1.3)

$$\frac{1}{2\mu}\frac{\partial}{\partial t}(\sigma_{22} - \sigma_{33}) + \frac{\partial \lambda}{\partial t}(\sigma_{22} - \sigma_{33}) = 0, \qquad \frac{1}{2\mu}\frac{\partial \sigma_{23}}{\partial t} + \frac{\partial \lambda}{\partial t}\sigma_{23} = 0 \tag{1.6}$$

When the state of stress ahead of the wave is described by $\sigma_{22} - \sigma_{33} = 0$ and $\sigma_{23} = 0$ (the last equation can be obtained by rotating the coordinate system about the Ox-axis), we have $\sigma_{22} - \sigma_{33} \equiv 0$ and $\sigma_{23} \equiv 0$. The case of $\sigma_{22} - \sigma_{33} = 0$, with the additional assumption of $v_3 = 0$ and $\sigma_{13} = 0$ was studied in [2]. Below we shall investigate the case $v_3 \neq 0$ and $\sigma_{13} \neq 0$, and the condition $\sigma_{22} - \sigma_{33} = 0$ will also be subsequently removed.

Performing the change of variables

$$J^2 = \frac{3}{4}(\sigma_{11} + p)^2 + \sigma_{12}^2 + \sigma_{13}^2, \quad \frac{\sqrt{3}}{2}(\sigma_{11} + p) = J\cos\theta \quad (0 \leqslant \theta < \pi)$$

$$\sigma_{12} = J\sin\theta\cos\varphi, \quad \sigma_{13} = J\sin\theta\sin\varphi \quad (0 \leqslant \varphi < 2\pi) \tag{1.7}$$

we obtain the system (1.3) - (1.5) in the plastic region in the form

$$\frac{\partial v_1}{\partial x} + \frac{1}{K}\frac{\partial p}{\partial t} = 0$$

$$-\frac{1}{2}\frac{\partial v_2}{\partial x} + \left(J\frac{d\Phi}{dJ} + \frac{1}{2\mu}\right)\frac{\partial J}{\partial t} + \frac{1}{2\mu}J\cos\theta\cos\varphi\frac{\partial \theta}{\partial t} - \frac{1}{2\mu}J\sin\theta\sin\varphi\frac{\partial \varphi}{\partial t} = 0$$

$$-\frac{1}{2}\frac{\partial v_3}{\partial x} + \left(J\frac{d\Phi}{dJ} + \frac{1}{2\mu}\right)\frac{\partial J}{\partial t} + \frac{1}{2\mu}J\cos\theta\sin\varphi\frac{\partial \theta}{\partial t} + \frac{1}{2\mu}J\sin\theta\cos\varphi\frac{\partial \varphi}{\partial t} = 0$$

$$\frac{1}{\sqrt{3}}\frac{\partial v_1}{\partial x} - \left(J\frac{d\Phi}{dJ} + \frac{1}{2\mu}\right) + \frac{1}{2\mu}J\sin\theta\frac{\partial \theta}{\partial t} = 0 \tag{1.8}$$

$$\rho_0\frac{\partial v_1}{\partial t} + \frac{\partial p}{\partial x} - \frac{2}{\sqrt{3}}\cos\theta\frac{\partial J}{\partial x} + \frac{2}{\sqrt{3}}J\sin\theta\frac{\partial \theta}{\partial x} = 0$$

$$\rho_0 \frac{\partial v_2}{\partial t} - \sin\theta\cos\varphi \frac{\partial J}{\partial x} - J\cos\theta\cos\varphi \frac{\partial\theta}{\partial x} + J\sin\theta\sin\varphi \frac{\partial\varphi}{\partial x} = 0$$

$$\rho_0 \frac{\partial v_3}{\partial t} - \sin\theta\sin\varphi \frac{\partial J}{\partial x} - J\cos\theta\sin\varphi \frac{\partial\theta}{\partial x} - J\sin\theta\cos\varphi \frac{\partial\varphi}{\partial x} = 0$$

We shall take J as the simple wave parameter in (1.8), i.e. we shall consider the solutions of (1.8) of the form $u = u\,[J\,(x,\,t)]$, where $u \equiv (v_1,\, v_2,\, v_3,\, J,\, p,\, \theta,\, \varphi)$. The system (1.8) which has the form $A\,(u)\;\partial u\,/\,\partial t + B\,(u)\;\partial u\,/\,\partial x = 0$, can be reduced to the following system of ordinary differential equations

$$[-\,C\,(u)\,A\,(u) + B\,(u)]\;\; du\,/\,dJ = 0 \qquad (1.9)$$

where $C\,(u)$ is the root of the characteristic equation $\det(-\,\mathrm{CA} + \mathrm{B}) = 0$

Investigation of the characteristic equation can be simplified by utilizing the Mandel theorem [4]

$$0 \leqslant C_1^p \leqslant C_1^e \leqslant C_2^p \leqslant C_2^e \leqslant C_3^p \leqslant C_3^e \qquad (1.10)$$

where C_i^e and C_i^p denote, respectively, the elastic and plastic characteristic velocities. Since

$$C_1^e = C_2^e = \sqrt{\mu\,/\,\rho_0}$$

we also have

$$C_2^p = \sqrt{\mu/\rho_0}$$

whereupon C_1^p and C_3^p can be found from the characteristic equation

$$\alpha a^4 + \beta a^2 + \gamma = 0 \qquad (a^2 \equiv \rho_0 C^2)$$

$$\alpha = J\frac{d\Phi}{dJ} + \frac{1}{2\mu},\; \beta = -J\frac{d\Phi}{dJ}\left(K + \mu + \frac{\mu}{3}\sin^2\theta\right) - \frac{K}{2\mu} - \frac{7}{6}$$

$$\gamma = \mu K J\frac{d\Phi}{dJ}\cos^2\theta + \frac{2}{3}\mu + \frac{1}{2}K \qquad (1.11)$$

The simple waves break up if $\partial C\,/\,\partial t = (dC\,/\,dJ)\,(dJ\,/\,\partial t) > 0$ or if $dC\,/\,dJ > 0$, since in the plastic region $\partial J\,/\,\partial t > 0$. If $dC\,/\,dJ < 0$, the simple wave becomes flatter. Under $d\,/\,dJ$ we understand a derivative given by (1.9). The expressions dC/dJ and da^2/dJ have the same sign for the waves moving to the right. Differentiating (1.11), we find that the sign of da^2/dJ coincides with the sign of $S \equiv a^4\; d\alpha/dJ + a^2\; d\beta\,/\,dJ + d\gamma\,/\,dJ$ for the waves moving at velocity C_1^p (slow), and is opposite for the waves moving at velocity C_3^p (fast)

$$S = \frac{d}{dJ}\left(J\frac{d\Phi}{dJ}\right)D + 2K\mu\left(1 + \frac{a^2}{3k}\right)J\frac{d\Phi}{dJ}\frac{d\theta}{dJ}\sin\theta\cos\theta \qquad (1.12)$$

$$D \equiv a^4 - (K + \mu + {}^1/_3\;\mu\;\sin^2\theta)\,a^2 + K\mu\;\cos^2\theta$$

In the following we assume that $d\,(Jd\,\Phi\,/\,dJ)\,/\,dJ > 0$. This inequality represents the condition of convexity of the stress-strain curve for the simple compression test (Fig.1). We shall show below that in this case no strong discontinuities are formed; when $d\,(J\;d\;\Phi\;/dJ)\;/\;dJ < 0$ the discontinuities obviously appear even when the motion has a single (longitudinal) velocity component.

From (1.9) we find

$$\frac{d\theta}{dJ} = \operatorname{ctg}\theta\left(\frac{1}{J} - \frac{2\mu\,(a^2 - K)}{K + 4\mu\,/\,3 - a^2}\,\frac{d\Phi}{dJ}\right)$$

Substituting a^2 from (1.11) and using (1.10) qe find that the signs of $d\theta\,/\,dJ$ and $\operatorname{tg}\theta$ are opposite for the fast waves, and equal for the slow waves. Further, by (1.11) we have

$$D = -\left(J\frac{d\Phi}{dJ}\right)^{-1}\left[\frac{1}{2\mu}a^4 - \left(\frac{K}{2\mu}+\frac{7}{6}\right)a^2 + \frac{2}{3}\mu + \frac{1}{2}K\right]$$

from where, in accordance with (1.10), it follows that $D \geqslant 0$ for the fast waves and $D \leqslant 0$ for the slow waves. Using (1.12) and the sign estimates obtained above we can finally establish that $dC \,/\, dJ \leqslant 0$ for both slow and fast plastic waves.

Thus, no shock waves form from the simple plastic waves. The waves which propagate at velocity

$$C_2^p = \sqrt{\mu / \rho_0}$$

without distorting their form can be regarded, in the limit (e.g. in a self-similar solution), as shock waves. For these waves the problem of determining the relations at the discontinuity is simple: all quantities vary as those in a simple wave. It should, however be noted that the result obtained follows from imposing arbitrarily strong restrictions. Break up of simple plastic waves is possible even within the framework of the geometrically linear theory with the effects of heat disregarded [1]. In this case additional conditions at the discontinuities can, and must be obtained by considering the structure of the discontinuity [5].

Finally, the condition $\sigma_{22} - \sigma_{33} = 0$ should be disposed of. If we have $\sigma_{22} - \sigma_{33} = \gamma_0$ ahead of the wave, then from (1.6) it follows that $\sigma_{22} - \sigma_{33} = \gamma_0 e^{-2\mu\lambda}$, and the equation of the stress surface

$$J^2 \equiv {}^3/_4(\sigma_{11}+p)^2 + \sigma_{12}^2 + \sigma_{13}^2 + {}^1/_4(\sigma_{22}-\sigma_{33})^2 = k^2[\chi(\lambda)] \quad \text{or} \quad \lambda = \Phi(J) \qquad (1.13)$$

can be rewritten in the form

$$I^2 \equiv {}^3/_4(\sigma_{11}+p)^2 + \sigma_{12}^2 + \sigma_{13}^2 = k^2[\chi(\lambda)] - {}^1/_4\gamma_0^2 e^{-4\mu\lambda} \quad \text{or} \quad \lambda = \Psi(I) \qquad (1.14)$$

Since $\sigma_{22}-\sigma_{33}$ does not enter the remaining equations of the system (1.3) – (1.5) the problem can be studied for a medium in which $\sigma_{22} - \sigma_{33} = 0$, while the equation of the stress surface has the form not of (1.13), but of (1.14). It remains to show that the relations $d\Phi \,/\, dJ > 0$ and $d\,(J\, d\Phi \,/\, dJ)\,/dJ > 0$ for (1.13) imply analogous relations $d\Psi \,/\, dI > 0$ and $d\,(I\, d\Psi \,/\, dI)\,/\,dI > 0$ for (1.14). The first of them is obvious, and the validity of the second one can be confirmed by differentiating (1.13) and (1.14) twice with respect to J and I, respectively, and using the inequality

$$\frac{d\Psi}{dI} = \frac{2I(\lambda)}{dk^2/d\lambda + \mu\gamma_0^2 e^{-4\mu\lambda}} < \frac{2J(\lambda)}{dk^2/d\lambda} = \frac{d\Phi}{dJ}$$

In particular, for ideally plastic media ($k^2 = \text{const}$) the problem with $\sigma_{22} - \sigma_{33} \neq 0$ is equivalent to that of a motion with $\sigma_{22} - \sigma_{33} = 0$ for a medium with restricted work-hardening property $I^2 = k^2 - {}^1/_4\, \gamma_0^2\, e^{-4\mu\lambda}$.

Below we study the simple waves for this case in detail and discuss the solution of the problem of collapse of a discontinuity.

2. Let us put in (1.1) $k^2 = \text{const}$, $v_3 \equiv 0$ and $\sigma_{13} \equiv 0$, but have $\sigma_{22} - \sigma_{33} \neq 0$. Performing the change of variables

$$\sigma_{12} = k\cos\theta \qquad (0 \leqslant \theta < \pi)$$

$$\sigma_{11} + p = \frac{2}{\sqrt{3}}k\sin\theta\cos\varphi, \quad \sigma_{22} - \sigma_{33} = 2k\sin\theta\sin\varphi \quad (0 \leqslant \varphi < 2\pi) \quad (2.1)$$

we can reduce the system (1.1), (1.3) and (1.5), in the present case, to

$$K\frac{\partial v_1}{\partial x} + \frac{\partial p}{\partial t} = 0, \quad \mu\frac{\partial v_2}{\partial x} - \frac{k}{\sin\theta}\frac{\partial\theta}{\partial t} - k\operatorname{ctg}\varphi\cos\theta\frac{\partial\varphi}{\partial t} = 0$$

$$\mu \frac{\partial p}{\partial t} + \frac{\sqrt{3}}{2} Kk \frac{\sin\theta}{\sin\varphi} \frac{\partial \varphi}{\partial t} = 0 \tag{2.2}$$

$$\rho_0 \frac{\partial v_1}{\partial t} + \frac{\partial}{\partial x}\left(p - \frac{2}{\sqrt{3}} k \sin\theta \cos\varphi\right) = 0$$

$$\rho_0 \frac{\partial v_2}{\partial t} + k \sin\theta \frac{\partial \theta}{\partial x} = 0, \quad 2\mu \frac{\partial \lambda}{\partial t} = -\operatorname{ctg}\theta \frac{\partial \theta}{\partial t} - \operatorname{ctg}\varphi \frac{\partial \varphi}{\partial t} \geqslant 0$$

The characteristic velocities C_+ and C_- of the system (2.2) are given by

$$a^4 - a^2(\mu \sin^2\theta + {}^4/_3 \mu \sin^2\varphi + {}^4/_3 \mu \cos^2\theta \cos^2\varphi + K) +$$
$$+ \mu \sin^2\theta \left(\frac{4\mu}{3} \sin^2\varphi + K\right) = 0, \qquad a^2 \equiv \rho_0 C^2 \tag{2.3}$$

and the inequalities hold for these velocities (1.10)

$$0 \leqslant C_- \leqslant \sqrt{\mu/\rho_0} \leqslant C_+ \leqslant \sqrt{(K + {}^4/_3 \mu)/\rho_0} \tag{2.4}$$

Use of θ as a parameter for the simple wave will be expedient. The system (2.2) can then be reduced to the following ordinary differential equations

$$\frac{d\varphi}{d\theta} = \frac{(\mu \sin^2\theta - a^2)\sin\varphi}{a^2 \sin\theta \cos\theta \cos\varphi} \tag{2.5}$$

$$\frac{K}{C}\frac{dv_1}{d\theta} = \frac{dp}{d\theta} = -\frac{\sqrt{3}}{2}\frac{Kk}{\mu}\frac{\sin\theta}{\sin\varphi}\frac{d\varphi}{d\theta}, \quad \frac{dv_2}{d\theta} = \frac{k\sin\theta}{\rho_0 C}$$

In order to investigate this system, it is obviously necessary to plot the integral curves of (2.5). Various cases are possible, depending upon which interval the quantity K / μ arrives at, the intervals being $(0,1)$, $(1, {}^4/_3)$ and $({}^4/_3, \infty)$. We shall assume for definiteness that $K / \mu > {}^4/_3$. The remaining cases can be treated in exactly the same way. The obvious symmetry implies that only the values $0 \leqslant \theta \leqslant \pi / 2$ and $0 \leqslant \varphi \leqslant \pi / 2$ and the simple waves propagating to the right need be considered.

We begin by considering slow simple waves. In this case Eq. (2.5) has the singular points $O(0, 0)$ and $K(0, \pi / 2)$ Expanding $a^2(\theta, \varphi)$ near the point O gives

$$a^2 = \frac{3K\mu}{2(3K + 2\mu)} \theta^2 + \theta^2 O(\theta^2 + \varphi^2) \tag{2.6}$$

Then from (2.4) it follows that near O

$$\frac{d\varphi}{d\theta} = \frac{4}{3} \cdot \frac{\mu}{K} \frac{\varphi}{\theta} \tag{2.7}$$

Point O is a node point and the integral curves touch the straight line $\theta = 0$. Similarly we find that the point K is a saddle point. The straight lines $\theta = \pi / 2$, $\varphi = 0$ and $\varphi = \pi \ \ 2$ are isoclinic of $d\varphi / d\theta = 0$ and $\theta = 0$ is an isoclinic of $d\varphi / d\theta = \infty$. Moreover, from (2.3) and (2.4) we find that

$$\mu \sin^2\theta \geqslant \rho_0 C_-^2 \tag{2.8}$$

and, in accordance with (2.5), we have in the region considered, $d\varphi / d\theta \geqslant 0$. Finally the pattern of the integral curves is shown in Fig. 2. The arrows show the direction in which the values change in a simple wave. The direction is determined by the inequality $\partial\lambda / \partial t > 0$, which reduces by virtue of (2.2) and (2.8) in the region considered to the condition

$$\partial\theta / \partial t < 0 \tag{2.9}$$

Let us inspect the changes which the remaining quantities undergo in the slow wave

$$1)\ \frac{dv_2}{d\theta} = \frac{k \sin\theta}{\rho_0 C_-} \geqslant 0$$

By (2.9) the variation $\Delta v_2 < 0$ and is bounded, since $dv_2 / d\theta$ tends according to (2.6) to unity when $\theta \to 0$.

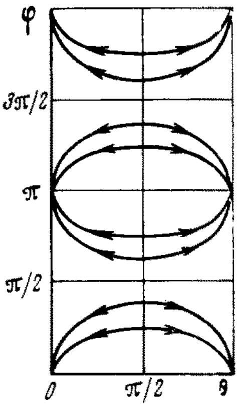

Fig. 2.

$$2)\ \frac{dv_1}{d\theta} = -\frac{\sqrt{3}}{2}\frac{k}{\mu}\frac{\sin\theta}{\sin\varphi}\frac{d\varphi}{d\theta}C_- \leqslant 0$$

The variation $\Delta v_1 > 0$ and is bounded, since by (2.6) and (2.7) $dv_1 / d\theta \to 0$ as $\theta \to 0$.

3) Finally, $\sigma_{12} = k \cos\theta$ and σ_{11} varies as v_1 since the equation of motion implies that $d\sigma_{11} / d\theta = -\rho_0 C\, dv_1 / d\theta$. The integral curve on the σ_{11}, σ_{12} -plane has the form $a\alpha$ (see Fig. 4 below).

It is clear that the value $|\sigma_{12}| = k$ can always be attained in a slow plastic wave. We also note that the changes undergone by all quantities in such a wave are restricted.

The fast simple waves are studied in an analogous manner. Figure 3 depicts the integral curves on the θ, φ-plane. By virtue of the inequality (2.4) we have $d\varphi / d\theta \leqslant 0$. The straight lines $\theta = 0$, $\theta = \pi / 2$ and $\varphi = \pi / 2$ are isoclinic of $d\varphi / d\theta = \infty$ and the line $\varphi = 0$ is an isoclinic of $d\varphi / d\theta = 0$. We have the following singularities: O which is a saddle point, $d\varphi / d\theta = -\varphi / \theta$ and M $(\pi / 2,\ 0)$ which is a node, the integral curves touch the line $\theta = \pi / 2$

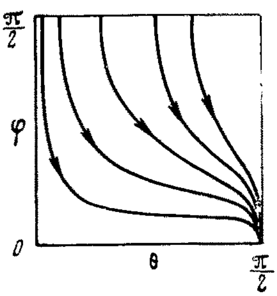

Fig. 3.

$$\frac{d\varphi}{d\theta_1} = \left(1 - \frac{\mu}{K}\right)\frac{\varphi}{\theta_1} \qquad \left(\theta_1 \equiv \theta - \frac{\pi}{2}\right) \qquad (2.10)$$

As in the previous case, all quantities in the fast wave vary monotonously. We find that $\Delta v_2 > 0$ and $\Delta\sigma_{12} < 0$ and are bounded as before, while v_1 and $-\sigma_{11}$ increase without bounds. Since $\varphi \to 0$ when $\theta \to \pi / 2$, by (2.10) we have

$$\frac{dv_1}{d\theta} = -\frac{\sqrt{3}}{2}\frac{k}{\mu}\frac{\sin\theta}{\sin\varphi}\frac{d\varphi}{d\theta}C_+ \sim -\frac{1}{\varphi}\frac{\varphi}{\theta_1} = -\frac{1}{\theta_1}$$

The integral curves on the σ_{11}, σ_{12} -plane have the form Aa (Fig. 4).

As was already shown in Sect 1, neither the fast nor the slow plastic waves do break up.

It remains to consider the simple plastic waves for which θ cannot be used as a parameter, i.e. the waves with $\sigma_{12} = \text{const}$. Let $\mu(x, t)$ be the parameter of such a wave. The second equation of (1.5) supplies an alternative: $dv_2 / d\mu = 0$ or $\partial\mu/\partial t = 0$, i.e. $C = 0$. In the first case (1.4) implies $\psi\sigma_{12} = 0$ ($\psi \equiv \partial\lambda / \partial t$). Setting $\psi = 0$ we obtain a solution which is a constant and $\sigma_{12} = 0$ represents the limiting case of a fast plastic wave which propagates without distorting its form when $\sigma_{22} = \sigma_{33}$ When $C = 0$ x can be used as a parameter of a simple wave. The system (1.1), (1.3), (1.5) then reduces to

$$\frac{1}{2}\frac{\partial v_2}{\partial x} = \psi\sigma_{12}, \quad \psi \geqslant 0, \quad \psi(\sigma_{11} + p) = 0$$
$$\psi(\sigma_{22} - \sigma_{33}) = 0, \quad {}^3/_4(\sigma_{11} + p)^2 + \sigma_{12}^2 + \frac{1}{4}(\sigma_{22} - \sigma_{33})^2 = k^2$$
$$\sigma_{11} = \text{const}, \qquad v_1 = \text{const}$$

and describes two types of solutions.

1) $\psi \neq 0$; σ_{ij} and v_1 are constant, $|\sigma_{12}| = k$; ${}^1/_2 \; dv_2/dx = \psi\sigma_{12}$ ($\psi > 0$ and is arbitrary elsewhere). The magnitude of v_2 in this solution varies arbitrarily and its sign is determined from the condition $\psi > 0$ by the sign of σ_{12}.

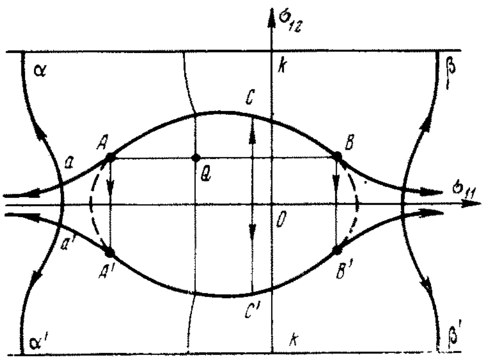

Fig. 4.

2) $\psi = 0$; σ_{11}, σ_{12}, v_1 and v_2 are constant, σ_{22} and σ_{33} vary arbitrarily.

In the limit (e.g. in the self-similar problem) these solutions reduce to two types of discontinuities encountered in the theory of quasi-static motions of a plastic medium [6]. In the first type of discontinuity pure shear occurs on both sides and the stresses are continuous, while at the second type discontinuity the velocities are continuous and the deformation velocities at both sides cancel each other ($\psi = 0$). Both discontinuities are of the contact type.

Other waves such as elastic longitudinal and transverse may appear in the medium under consideration in addition to the plastic waves, and their parameters vary in the following, well-known manner; for the longitudinal wave we have

$$\rho_0 C_\parallel^2 = K + \frac{4}{3}\mu, \quad \Delta\sigma_{11} = -\rho_0 C_\parallel \Delta v_1, \quad \Delta\sigma_{22} = \Delta\sigma_{33} =$$
$$= \frac{-3K + 2\mu}{3K + 4\mu}\rho_0 C_\parallel \Delta v_1, \quad \Delta\sigma_{12} = 0, \quad \Delta v_2 = 0 \tag{2.11}$$

and for the transverse wave we have

$$\rho_0 C_\perp = \mu, \ \Delta\sigma_{11} = \Delta\sigma_{22} = \Delta\sigma_{33} = 0, \ \Delta v_1 = 0, \ \Delta\sigma_{12} = -\rho_0 C_\perp \Delta v_2 \tag{2.12}$$

Let us investigate the solution to the problem of collapse of an arbitrary discontinuity. Suppose the values of σ_{ij} and v_i are given at $t = 0$ as s_{ij}^+ and u_i^+ for $x > 0$ and as s_{ij} and u_i for $x < 0$ ($u_i^+ < u_i$). All constants entering the equations and initial conditions of the problem have the dimension of velocity, density or stress; therefore a single dimensionless combination can be formed from x and t (e.g. $xt^{-1}(K/\rho_0)^{-1/2}$) and the problem is self-similar.

In the region $x > 0$ the self-similar solution consists of elastic shock waves and simple plastic waves propagating to the right and separating from each other by regions in which all parameters have constant values. The order of the wave propagation is determined by the inequality (2.4).

Let the initial state for $x > 0$ be elastic and represented on the σ_{11}, σ_{12}-plane by the point Q (Fig. 4). Points lying on the segment AB can be reached in the longitudinal elastic wave, and the points A and B themselves correspond to the emergence at the stress surface. On the θ, φ-plane the points corresponding to A and B are symmetric with respect to the straight line $\varphi = \pi/2$ ($\varphi = 3\pi/2$), since by (2.1) and (2.11) we have

$$k \cos \theta(A) = \sigma_{12}(A) = \sigma_{12}(B) = k \cos \theta(B)$$

$$2k \sin\theta \sin\varphi(A) = \sigma_{22}(A) - \sigma_{33}(A) = \sigma_{22}(B) - \sigma_{33}(B) = 2k \cos\theta \sin\varphi(B)$$

The longitudinal elastic wave may be followed by a fast plastic wave (curves Aa and Bb in Fig. 4). As was shown before, σ_{11} may vary in this wave without restrictions.

The states represented in Fig. 4 by the point of the curve $aAQBb$, i.e. after the passage of the fast plastic and longitudinal elastic waves, may be traversed by a transverse elastic wave in which we reach the states represented by the elliptic arc ACB (*)

$$\sigma_{12}^2 + \frac{3}{4}\left(\frac{3K}{3K+4\mu} s_{11}^+ - \frac{1}{3} s_{kk}^+ + \frac{4\mu}{3K+4\mu}\sigma_{11}\right)^2 = k^2 - \frac{1}{4}(s_{22}^+ - s_{33}^+)^2 \qquad (2.13)$$

and the curve $a'A'C'B'b'$ symmetrical with respect to $aACBb$ relative to the straight line $\sigma_{12} = 0$. The formula (2.13) is obtained from the condition of plasticity with the help of (2.11) and (2.12), the curves $aACBb$ and $a'A'C'B'b'$ are symmetric since in the transverse wave σ_{12} is the only stress that varies and the initial and final points (e.g. a and a') lie on the load surface.

In what follows, the propagation will be limited to the slow plastic waves (the curves $a\alpha$, $a'\alpha'$, $b\beta$ and $b'\beta'$ in Fig. 4) in which, as was shown before, the value of $|\sigma_{12}| = k$ is attained. The state of stress with arbitrary σ_{11} and σ_{12}, $|\sigma_{12}| \leqslant k$ (the last inequality is dictated by the condition $^1/_2\, \sigma_{ij}'\sigma_{ij}' = k^2$) can be reached from any prescribed initial state and this yields a solution to the problem on an oblique shock, i.e. to the problem in which a load σ_{11} and σ_{12} is applied to the surface $x = 0$ at the time $t = 0$ and remains constant henceforth. This problem was studied in [1, 2 and 7] under the condition $\sigma_{22} - \sigma_{33} = 0$

To construct a solution to the problem on collapse of the discontinuity we must inspect the variation of v_i from the given initial state s_{ij} and u_i along each path on the σ_{11}, σ_{12}-plane. Let the following relation hold for the waves propagating to the right

$$v_1^+ = u_1^+ + f_1(\sigma_{11}, \sigma_{12}, s_{ij}^+), \quad v_2^+ = u_2^+ + f_2(\sigma_{11}, \sigma_{12}, s_{ij}^+) \qquad (2.14)$$

Then for the waves moving to the left we have

$$v_1 = u_1 - f_1(\sigma_{11}, \sigma_{12}, s_{ij}), \quad v_2 = u_2 - f_2(\sigma_{11}, \sigma_{12}, s_{ij}) \qquad (2.15)$$

The variation of velocity is not restricted in the fast plastic wave, and is restricted in the remaining waves. Hence

$$\lim_{\sigma_{11}\to\infty} f_1(\sigma_{11}, \sigma_{12}, s_{ij}) = -\infty, \qquad \lim_{\sigma_{11}\to-\infty} f_1(\sigma_{11}, \sigma_{12}, s_{ij}) = +\infty$$

Then from (2.14) and (2.15) it follows that for any s_{ij} and u_i on the σ_{11}, σ_{12}-plane and for any value of σ_{12} a point exists in which $v_1^+ = v_1$. These points form a curve Γ_1 on the σ_{11}, σ_{12}-plane.

*) If the initial state Q lies on the load surface, it is represented by one of the points A, A', B, B' lying on the corresponding ellipse.

In certain cases (e.g. when the initial discontinuity is sufficiently small and s_{ij}^{+} as well as s_{ij}^{-} lie within the elastic region), a curve Γ_2 also exists, intersecting Γ_1, on which $v_2^+ = v_2^-$. The point P of intersection of Γ_1 and Γ_2 gives a solution of the problem of collapse of a discontinuity consisting of integral curves connecting, on the σ_{11}, σ_{12}-plane the points (s_{11}^{+}, s_{12}^{+}) and (s_{11}^{-}, s_{12}^{-}) with P, and a contact (second) type discontinuity discussed previously. The only quantities which undergo a jump are σ_{22} and σ_{33}.

The curves Γ_1 and Γ_2 need not intersect. The curve Γ_2 does not even always exist Since the variation of v_2 in the elastic and plastic (fast and slow) waves is restricted, v_2^+ and v_2^- do not coincide when $|u_2^+ - u_2^-|$ is sufficiently large. In this case we have, at the surface $x = 0$ a contact discontinuity of the first type. The states of stress on the σ_{11}, σ_{12}-plane at both sides of such a discontinuity are represented by the intersections of Γ_1 with $\sigma_{12} = k$ or with $\sigma_{12} = -k$. The condition $\operatorname{sgn} \sigma_{12} = \operatorname{sgn} (v_2^+ - v_2^-)$ enables us to choose this point unambiguously. Since Γ_1 does not intersect Γ_2, the last sign along Γ_1 is retained. Connecting the selected point with the initial points by means of the integral curves, we obtain a solution to the problem of collapse of the initial discontinuity.

The jump suffered by the value of the transverse velocity component is characteristic for the ideally plastic media, and the media with restricted work-hardening property. A situation discussed in [2] is typical for the media with unrestricted work-hardening property. Numerical calculations in [2] indicate that in a slow plastic wave $\sigma_{12} \to \infty$, $\sigma_{11} - \sigma_{22} \to 0$ and σ_{11} is restricted. (These conclusions could easily be reached in a qualitative manner, as one of the equations of the system becomes separated from the other equations just as in the example discussed previously). Estimating the terms of the characteristic equations we then find that $C_- \sim \sigma_{12}$ Choosing σ_{12} as the parameter of the simple wave we find from the equation of motion that in the slow wave $dv_2 / d\sigma_{12} = -(\rho_0 C_-)^{-1} \sim \sigma_{12}$ and $v_2 \to \infty$ in the slow wave. Apparently in this case v_2 can always be made continuous at the contact discontinuity.

Thus we find that under the assumption made about the medium and the class of the problems considered, an arbitrary discontinuity decomposes into elastic shock waves, simple plastic waves and a contact type discontinuity. No discontinuities in the plastic region exist that would require additional conditions to be obtained. The fact that the simple plastic waves do not break up in the more general case discussed in Sect. 1 enables one to conclude with sufficient confidence that in the present case the situation will remain exactly the same and the only discontinuities that need to be considered will be the elastic shock waves, the contact type discontinuities and discontinuities representing limiting cases of the simple waves propagating with constant speed.

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ASYMPTOTIC METHODS OF SOLVING NONLINEAR PARTIAL DIFFERENTIAL EQUATIONS

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The asymptotic method presented here for one-dimensional nonlinear dynamic systems described in terms of partial differential equations with a small parameter, uses a known solution of the unperturbed problem as the basis for constructing an approximate solution on the prescribed variable range, which will tend to its exact value when the small parameter tends to zero. The method is based essentially on varying the arbitrary constants entering the unperturbed solution and constructing, for the slowly varying functions of the coordinate and time thus created, a system of differential equations the form of which depends on the degree of approximation. These equations remain nonlinear in the partial derivatives thus retaining the specific character of the problem and are, at the same time, easier to analyze than the initial equations.

The substantiation of the method is reduced to proving a theorem on continuous dependence of the solution of the system of partial differential equations on the variation of its right-hand sides, and the proof is given here for hyperbolic and symmetrical parabolic systems.

The procedure considered here embraces, as its particular cases, the known asymptotic methods of the perturbation theory [1, 2] of the geometrical optics [3, 4] and the methods [5, 6] related to the method for ordinary differential equations which are almost linear [7].

1. Let us consider a system of differential equations of the form

$$N(u) = u_t + A(u, x, t, \chi, \tau)\, u_x + B(u, x, t, \chi, \tau) =$$